# STOCHASTIC APPROXIMATION APPROACHES FOR DISCRETE-TIME NONLINEAR STOCHASTIC OPTIMAL CONTROL PROBLEM IN ENGINEERING APPLICATIONS

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Dedicated to my respected supervisor, Dr. Kek Sie Long,
who guided and led me during the study.

To my beloved parents,
who gave birth of me and always accompany me,
and guiding and supporting me in all situations.

Also to my family members and my friends,
who have always been there for me and have made my life more colourful.

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## ABSTRACT

Decisions and control of stochastic dynamical systems are challenging tasks. This thesis explores the use of the stochastic approximation (SA) approach to solve discrete-time nonlinear stochastic optimal control problems in engineering. In the presence of Gaussian white noise, the state dynamics become fluctuate, uncertain and incomplete information. So, optimizing and controlling such dynamic systems will not provide a satisfactory solution. Therefore, the SA for state-control (SASC) algorithm is proposed to associate state estimation and control law design for solving the control problem. Then, the optimal solution of the extended Kalman filter (EKF) is compared as a benchmark solution. Moreover, the variants of the SA approach, namely SA with momentum (SAM), Nesterov accelerated gradient (NAG), and adaptive moment estimation (Adam), are applied in the SASC algorithm for better iterations. For illustration, engineering applications, which are inverted pendulum-cart system, four-tank system, and Duffing electrical oscillator, are studied. The simulation results showed that trajectories of state and output are estimated close to actual trajectories using the optimal control law designed. From these results, the tilt angle and the cart position were regulated around steady states through the optimal external force. In addition, the liquid levels in four tanks were optimally estimated upon the optimal voltages of pumps. Further, the flux and voltage of the nonlinear inductor were optimally calculated under the sinusoidal source voltage. The efficiency and accuracy of the SASC algorithm with Adam are highly recommended. In conclusion, the SASC algorithm is applicable for solving discrete-time nonlinear stochastic optimal control problems effectively.

# ABSTRAK

Pembuatan keputusan dan kawalan bagi sistem berdinamik stokastik adalah tugas yang mencabar. Tesis ini meneroka penggunaan kaedah penghampiran stokastik (SA) untuk menyelesaikan masalah kawalan optimum stokastik tak linear masa diskrit dalam kejuruteraan. Dengan kehadiran bunyi putih Gaussian, dinamik keadaan menjadi turun naik, tidak pasti dan maklumat tidak lengkap. Jadi, pengoptimuman dan kawalan sistem dinamik tersebut tidak akan memberikan penyelesaian yang memuaskan. Oleh itu, algoritma SA bagi keadaan-kawalan (SASC) dicadangkan supaya menggabungkan anggaran keadaan dan rekabentuk hukum kawalan untuk menyelesaikan masalah kawalan. Kemudian, penyelesaian optimum penapis Kalman lanjutan (EKF) diperbandingkan sebagai penyelesaian penanda aras. Selain itu, varian kaedah SA, iaitu SA bermomentum (SAM), kecerunan dipercepatkan Nesterov (NAG), dan anggaran momen penyesuaian (Adam), digunakan dalam algoritma SASC demi lelaran yang lebih baik. Sebagai penerangan, penggunaan kejuruteraan, iaitu sistem kereta-bandul terbalik, sistem empat tangki, dan pengayun elektrik Duffing, dikaji. Keputusan simulasi menunjukkan bahawa trajektori keadaan dan keluaran dianggarkan hampir dengan trajektori sebenar menggunakan hukum kawalan optimum yang direkabentuk. Dari keputusan ini, sudut kecondongan dan kedudukan kereta dapat dikawal di sekitar keadaan mantap melalui daya luaran yang optimum. Di samping itu, paras cecair dalam empat tangki dianggarkan secara optimum berdasarkan voltan optimum pam. Selanjutnya, fluks dan voltan induktor tak linear dikira secara optimum dengan voltan berpunca sinusoidal. Kecekapan dan ketepatan algoritma SASC dengan Adam amat dicadangkan. Kesimpulannya, algoritma SASC boleh digunakan untuk menyelesaikan masalah kawalan optimum stokastik tak linear masa diskrit dengan berkesan.

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| | | |
|---|---|---|
| $A$ | - | State transition matrix |
| $A_i$ | - | Area of cross-section of Tank $i$ |
| $AC$ | - | Sinusoidal source (alternating current) |
| $a_i$ | - | Gain sequence / Cross-section of the outlet pipe of Tank $i$ |
| $B$ | - | Control-input matrix |
| $\beta$ | - | Hyperparameter |
| $C$ | - | Capacitor / Measurement matrix |
| $E[\cdot]$ | - | Expectation operator |
| $E$ | - | Source voltage |
| $\varepsilon$ | - | Tolerance |
| $F$ | - | Externally force |
| $f$ | - | Plant function |
| $\varphi$ | - | Terminal cost |
| $G$ | - | Noise coefficient matrix |
| $g$ | - | Stochastic gradient / Acceleration of gravity |
| $\gamma$ | - | Momentum |
| $\gamma_i$ | - | Flow parameter of Pump $i$ |
| $H$ | - | Hamiltonian function |
| $h$ | - | Output measurement channel |
| $\eta$ | - | Learning rate / Measurement noise vector |
| $i$ | - | Iteration number / Electrical current |
| $i_R$ | - | Current of the resistor |
| $i_C$ | - | Current of the capacitor |
| $J$ | - | Expected cost function |
| $J'$ | - | Augmented cost function |

| | | |
|---|---|---|
| $J_{sse}$ | - | Sum of squares errors |
| $K_f$ | - | Kalman filter gain |
| $k$ | - | Discrete time step |
| $k_i$ | - | Constant of Pump $i$ |
| $L$ | - | Cost under summation / Lagrangian |
| $l$ | - | Length of the pendulum rod |
| $l_i$ | - | Liquid level of Tank $i$ |
| $M$ | - | Cart mass |
| $M_0$ | - | State error covariance matrix |
| $M_x$ | - | Predicted state error covariance matrix |
| $m$ | - | Estimate of the mean / ballpoint mass |
| $\hat{m}$ | - | Bias-corrected estimate of the mean |
| $N$ | - | Final time step |
| $N_L$ | - | Nonlinear inductor |
| $\phi$ | - | Flux over the nonlinear inductor |
| $P$ | - | Error covariance |
| $p$ | - | Costate sequence |
| $\rho$ | - | Constant |
| $Q$ | - | State weighting matrix |
| $Q_\omega$ | - | Process noise covariance matrix |
| $q_{ini}$ | - | Flows into Tank $i$ |
| $q_{pumpi}$ | - | Flows out of the electrical Pump $i$ |
| $\theta$ | - | Tilt angle |
| $R$ | - | Resistor / Measurement weighting matrix |
| $R_\eta$ | - | Measurement noise covariance matrix |
| $\Re$ | - | Set of real numbers |
| $S$ | - | Weighting matrix |
| $t$ | - | Time |
| $T$ | - | Kinetic energy |
| $\tau$ | - | Sampling time |

| | | |
|---|---|---|
| $u_i$ | - | Input variable at time $i$ |
| $u$ | - | Control vector |
| $u_0$ | - | Initial control |
| $V$ | - | Potential energy |
| $V_L$ | - | Voltage of the inductor |
| $V_R$ | - | Voltage of the resistance |
| $v$ | - | Velocity / Estimate of the unscented variance |
| $v_i$ | - | Input voltages to Pump $i$ |
| $\hat{v}$ | - | Bias-corrected estimate of the unscented variance |
| $\omega$ | - | External force frequency / Process noise vector |
| $x$ | - | State vector / Cart position along the $x$-direction |
| $\hat{x}$ | - | Optimal state estimate |
| $\bar{x}$ | - | Predicted state estimate |
| $x_0$ | - | Initial state |
| $\bar{x}_0$ | - | Mean of initial state |
| $x_d$ | - | State variable of Duffing electrical oscillator model |
| $x_f$ | - | State variable of four-tank system |
| $x_m$ | - | $x$-coordinate of the time-dependent center of gravity |
| $x_p$ | - | State variable of inverted pendulum-cart system |
| $y$ | - | Output measurement vector |
| $y_d$ | - | Output variable of Duffing electrical oscillator model |
| $y_f$ | - | Output variable of four-tank system |
| $y_i$ | - | Output variable at time $i$ |
| $y_m$ | - | $y$-coordinate of the time-dependent center of gravity |
| $y_p$ | - | Output variable of inverted pendulum-cart system |
| $\tilde{y}$ | - | Measurement residual |
| $\hat{y}$ | - | Output estimate |
| AdaGrad | - | Adaptive gradient |
| Adam | - | Adaptive moment estimation |

| | | |
|---|---|---|
| AMSE | - | Asymptotic mean square error |
| B.C. | - | Before Christ |
| CNN | - | Convolutional neural network |
| DCNN | - | Deep convolution neural network |
| DNNs | - | Deep neural networks |
| DTN | - | Delay tolerant networks |
| EKF | - | Extended Kalman filter |
| FMMRAC | - | Fuzzy modified model reference adaptive control |
| i.i.d. | - | Independent and identically distributed |
| IAE | - | Integral absolute error |
| ILC | - | Iterative learning control |
| ISE | - | Integral square error |
| KF | - | Kalman filtering |
| LMI | - | Linear matrix inequality |
| LQG | - | Linear quadratic Gaussian |
| LQR | - | Linear quadratic regulator |
| MIMO | - | Multiple-input multiple-output |
| MLlib | - | Machine learning library |
| MPC | - | Model predictive control |
| MSE | - | Mean square errors |
| NAG | - | Nesterov accelerated gradient |
| PI | - | Proportional-integral |
| PID | - | Proportional-integral derivatives |
| RDSA | - | Random directions stochastic approximation |
| RMSProp | - | Root mean square propagation |
| RNNs | - | Recurrent neural networks |
| SA | - | Stochastic approximation |
| SASC | - | Stochastic approximation for state-control |
| SAM | - | Stochastic approximation with momentum |
| SGD | - | Stochastic gradient descent |
| SSE | - | Sum squares of errors |
| UKF | - | Unscented Kalman filter |
| ZN | - | Ziegler Nichols |

# CHAPTER 1

# INTRODUCTION

## 1.1 General introduction

Optimal control theory is a branch of applied mathematics and control engineering that uses mathematical optimization techniques to determine the optimal values of a set of control variables for minimizing the performance index over a dynamic system (Kirk, 2004). In a control problem, a cost function consists of the state and control variables, and a dynamic system is a class of first-order differential equations (La Torre *et al.*, 2015). The main aim of solving the control problem is to optimize the cost function under the control efforts in which the system evolves toward stabilization. In recent years, optimal control has become a well-known research frontier area. Its contributions to real-world problems, both for deterministic and stochastic cases (Fleming and Rishel, 2012) in engineering, biology, medicine, finance, ecology, economics, and management, have been recognized.

A discrete-time system is a signal processing entity that processes the discrete-time signal (Baltar and Nossek, 2014). In the system, both the input and output are discrete-time signals. According to Neishtadt (2007), discrete-time is referred to the time in a set of integers. A dynamical system is considered nonlinear if it does not obey the superposition principle (Saat *et al.*, 2017), which means its output is not strictly proportional to its input. In the real world, most systems are nonlinear. However, nonlinear systems are more difficult to analyse as they cannot be decomposed and solved independently.

Moreover, optimization and control of a dynamical system, which is disturbed by random noises, are very challenging tasks. This is because in the presence of the random effect of noises, the fluctuation behaviour arising would lead the dynamical system to an undesired solution. This issue has commonly happened in the real world, especially in engineering areas. The problem in such a stochastic dynamic system is commonly known as the stochastic optimal control problem (Kappen, 2008).

In engineering applications, optimal control ensures a strategic approach to increase productivity and enhance the best practice of operations in engineering systems. Applying the optimal control in engineering can minimize redundant manual controls and reduce human errors that require large expenses. It also compensates for random disturbance, allowing engineering systems to produce a correct output even in the presence of disturbance (Nise, 2020). Therefore, the formulation of a mathematical model for studying stochastic systems is very crucial for control and decision-making problems in engineering.

In particular, the linear quadratic Gaussian (LQG) model, which is a common mathematical model for studying the linear stochastic optimal control problem, is widely applied in dealing with real-world problems. The structure of the LQG model reveals the combination of the linear quadratic regulator (LQR) model and the Kalman filtering (KF) theory (Huerta *et al.*, 2011) that provides a fundamental theory for solving the linear stochastic optimal control problem. Due to its simplicity, the KF theory is one of the most generally used approaches for tracking and estimation (Julier and Uhlmann, 1997). The KF theory has been the focus of extensive application and research since the publication of the famous paper by Kalman, which describes a recursive solution to the discrete data of the linear filtering problem (Kalman, 1960b). Since then, it has been widely used in many areas, particularly in autonomous navigation, which is largely upon the advancement of digital computing.

## 1.2 Background of study

The optimal control theory has a long history of more than 360 years (Sargent, 2000). However, it took off after the achievement of optimal trajectory prediction in aerospace applications in the early 1960s. In 1638, Galileo introduced two shaped problems, namely the catenary (Conti *et al.*, 2017) and the brachistochrone (Nishiyama, 2013). Newton solved these shaped problems for the first time in 1685, but the results did not publish until 1694. In 1696, Johann Bernoulli challenged his colleagues to solve the brachistochrone problem, but Bernoulli published the solution in April 1697 (Herrera, 1994).

The competition sparked the interest of mathematicians in solving the problems of the catenary and the brachistochrone. The resulting ideas were collected in a book and Euler published this book in 1744. Euler generalized the problem as finding a curve within a given interval to minimize the cost function and the necessary conditions for optimality were provided (Fraser, 2005). Later, in 1755, Lagrange provided an analytical method based on the changes in the optimal curve, which led directly to the necessary conditions that were proposed by Euler. Thus, the result of the necessary condition was named the Euler-Lagrange equation.

After that Legendre investigated the second variation in 1786. At the same time, Hamilton reconstructed the equation and introduced the function as the Hamiltonian function. In 1838, Jacobi introduced the Hamilton-Jacobi equation, which served as the basis of dynamic programming developed by Bellman over a century later (Sussmann and Willems, 1997). On the other hand, Weierstrass also discovered the excess function, which is the forerunner of the maximum principle of Bellman and Pontryagin.

In 1957, Bellman proposed a new perspective on Hamilton-Jacobi theory called dynamic programming (Dreyfus, 2002). While McShane and Pontryagain extended the calculus of variations to handle control inequality constraints. In the 1950s, Pontryagin outlined the necessary conditions for optimality in his famous principle of maximum. Later, Rudolf Kalman developed the formulation of the LQR and the KF in the 1950s (Kalman, 1960a; Bryson, 1996) to establish the modern control era.

Stochastic optimal control is one of the active research areas in the control theory that deals with the existence of uncertainty in the observation and the randomness of noise disturbance in the dynamic system (Ahmed, 1973; Ahmed and Teo, 1974). The problem of stochastic optimal control is described as determining a set of admissible control to minimize an expected cost function over a general class of differential equations or difference equations in the presence of random disturbances (Adomian, 1985). In this situation, the random noise, which is known as the Gaussian white noise, affects the evolution of dynamic systems and the observation of state trajectories.

The state estimation, which is conducted by using the KF approach, is an important step in handling the stochastic dynamical system. By considering the errors of state and output, the KF approach provides the optimal state estimate for the linear dynamic systems, while the extended KF (EKF) approach is the common method used for estimating the state of nonlinear dynamic systems (Bryson and Ho, 1975). In other words, the state estimation problem can be defined as a stochastic optimization problem over the stochastic dynamic system. The aim is to obtain the optimal state estimate in which the sum of squares errors of state and output are minimized. With the state estimate, the optimal control law (Lewis *et al*., 2012) can be designed to minimize the performance index of the dynamic system.

On the other hand, the stochastic approximation (SA) approach (Spall, 2005), which is also known as the stochastic gradient descent (SGD) method, is an efficient method for solving stochastic optimization problems. It was first proposed by Robbins and Monro (1951), and then Kiefer and Wolfowitz (1952) released a paper to discuss the use of the SA to the regression problem. Since the gradient descent, which was invented by Cauchy in 1947, converges to a local minimum quite slowly, this limitation can be addressed by using the SGD method. Nowadays, the SGD method is recognized as one of the famous optimization approaches in machine learning.

Recently, the SA approach has been the central efficient and effective optimization method in machine learning, such as deep learning, supervised learning, unsupervised learning, and reinforcement learning (Sun *et al*., 2019). Nevertheless, the applicability of the SA approach in handling stochastic optimal control problems shall be further investigated as it can be used to minimize errors between the predicted results and the actual observation. Hence, in our study, the SA approach will be applied to estimate the state of dynamical systems, in turn, to design the optimal control law for solving discrete-time nonlinear stochastic optimal control problems.

Moreover, some recent variants of the SA approach have been developed for solving stochastic optimization problems. The SA with momentum (SAM) replaces the current gradient with momentum, which is an aggregate of the gradient (Karim, 2018), to update the weight instead of relying solely on the current gradient. Then, a similar update is implemented using Nesterov accelerated gradient (NAG), where the projected gradients are used. Subsequently, the adaptive moment estimation (Adam) that computes adaptive learning rates for each parameter is proposed. So, applying these variants of the SA approach, we want to identify the accuracy of these methods for state estimation and the efficiency of the optimal control design when solving discrete-time nonlinear stochastic optimal control problems.

## 1.3 Statement of problem

In the presence of Gaussian white noise in dynamic systems, the state dynamics become fluctuate, and the entire state trajectory is uncertain. This behaviour arises obviously in stochastic optimal control problems. Since the Gaussian white noise has zero mean and finite variance, it is a common noise consideration in the simulation of stochastic optimal control problems compared with colour noise, which the mean and variance are unknown. When taking the expectation, the state propagation seems to be a deterministic state equation. However, without the complete state information, optimizing and controlling a dynamic system will not provide a satisfactory solution, and even the exact solution is impossible to obtain. Therefore, using the appropriate computational technique for solving the stochastic optimal control problem is becoming a must-use tool, especially under incomplete or partially complete state information (Lewis *et al*., 2012). Thus, an efficient computational approach shall be developed and proposed for solving the stochastic optimal control problem from a practical perspective.

In past studies, the KF method has been applied for navigation, tracking, and estimation because of its simplicity and tractable. However, the application of the KF method to a nonlinear system can be more challenging. Since the KF method assumes that the system and observation model equations are both linear, this is not realistic in many real-life situations. In a nonlinear system, the linearization procedure is usually needed in deriving a filtering algorithm. The most common approach is to use the EKF approach. But the EKF approach requires the first-order derivative for the process model and the output model, which is costly, difficult to implement, and only reliable for systems that are almost linear on the time scale of the update intervals (Julier and Uhlmann, 1997). Moreover, using the nonlinear filtering theory to estimate the state dynamics would be computationally demanding and the design of the optimal control law is less accurate in a practical sense (Ahmed and Teo, 1974). Hence, an efficient computational algorithm is required to resolve these weaknesses.

Therefore, the study in this thesis aims to propose an efficient computational method of the SA for estimating the state dynamic and designing the optimal control in solving the discrete-time nonlinear stochastic optimal control problem. In our study, the iterative algorithm, which is called the SA for state-control (SASC) algorithm, is proposed for the state estimation and control law design. This algorithm only requires the initial value of the state error covariance matrix, unlike the KF approach needs to derive the equation of the state error covariance matrix. The SASC algorithm will include the SA variants, which are SAM, NAG, and Adam approaches, for handling nonlinear stochastic optimal control problems, and their accuracy and efficiency will be examined.

## 1.4 Objectives of study

The objectives of the study in this thesis are given as follows:

(a) To propose the SASC algorithm for estimating the state dynamic and designing the control law for solving discrete-time nonlinear stochastic optimal control problems.

(b) To compare the accuracy of the SASC algorithm with the EKF algorithm through mean square errors for solving discrete-time nonlinear stochastic optimal control problems.

(c) To verify the efficiency of the SA variants, namely SAM, NAG and Adam approaches, in the SASC algorithm for solving discrete-time nonlinear stochastic optimal control problems.

## 1.5 Scope of study

In this study, an efficient computational approach of SA, which is called the SASC algorithm, is discussed for solving stochastic optimal control problems in engineering applications. The SA approach and the recent variants of the SA approach are further investigated to carry out the state estimation and to design the optimal control law. These SA approach variants are SAM, NAG and Adam. Three engineering application examples, which are the inverted pendulum-cart system, the four-tank system, and the Duffing electrical oscillator model, are studied for illustration. These problems are defined as discrete-time nonlinear stochastic optimal control problems and are solved by using the SASC algorithm. The accuracy and efficiency of the SASC algorithm are compared with the EKF algorithm for verification.

## 1.6 Significance of study

This study is important and useful in various disciplines, particularly in engineering applications. Here, the engineering application is defined as the applied mathematical model that shows practical usage in engineering. The efficient computational approach, which satisfies the certainty equivalence property, is proposed in this study to solve discrete-time nonlinear stochastic optimal control problems. The methodology of the computational approach is discussed and the applicability of the computational approach in engineering applications is verified. In summary, the following contributions are aimed:

(a) An efficient computational approach for solving nonlinear stochastic optimal control problems is proposed, where the SA approach is employed. This computational approach is named the SASC algorithm.

(b) An improvement in the design of the control law through the SA updating rule is carried out for incorporating the state estimation.

(c) The illustrative examples of the stochastic optimal control problem in engineering applications are studied, where the accuracy and efficiency of the SASC algorithm and its variants are proven.

Here, the outcome of this study is expected to provide the optimal decision strategy for stochastic optimal control problems, which is very useful in many disciplines, particularly in engineering. For this purpose, three examples, which are the inverted pendulum cart system, four-tank system, and Duffing electrical oscillator model, are illustrated.

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